Global Optimization by Bound Contraction

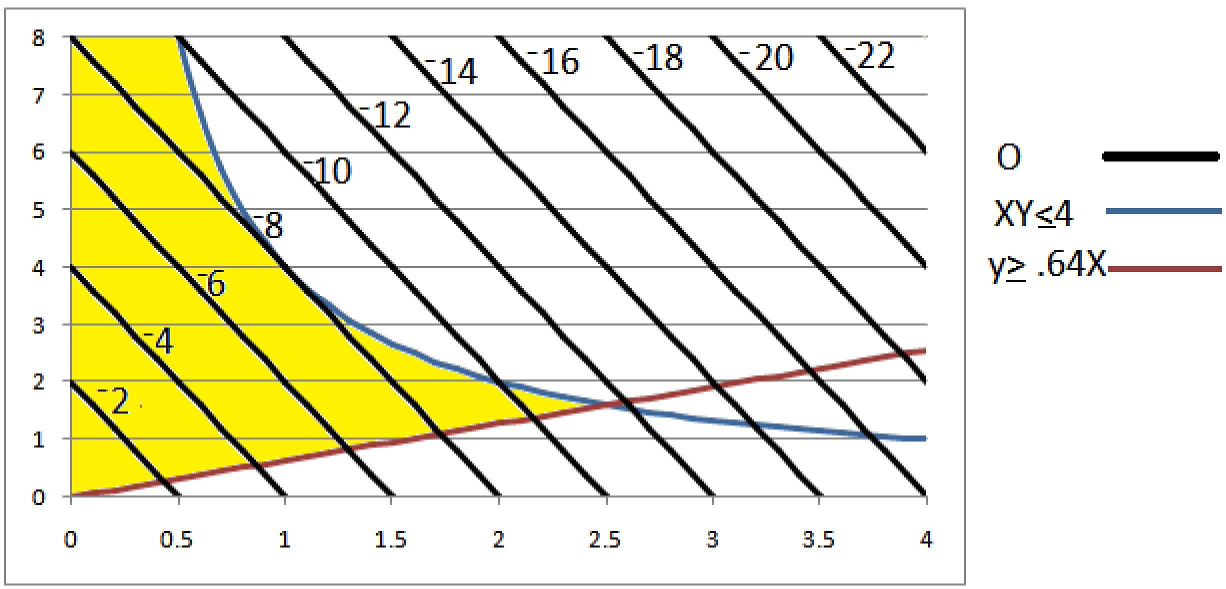
Function to Optimize:

minimize O=-4X-Y

Subject To:

$0 \le X \le 4$
$0 \le Y \le 8$
$X \cdot Y \le 4$
$0.64X \le Y$

UB: -11.6



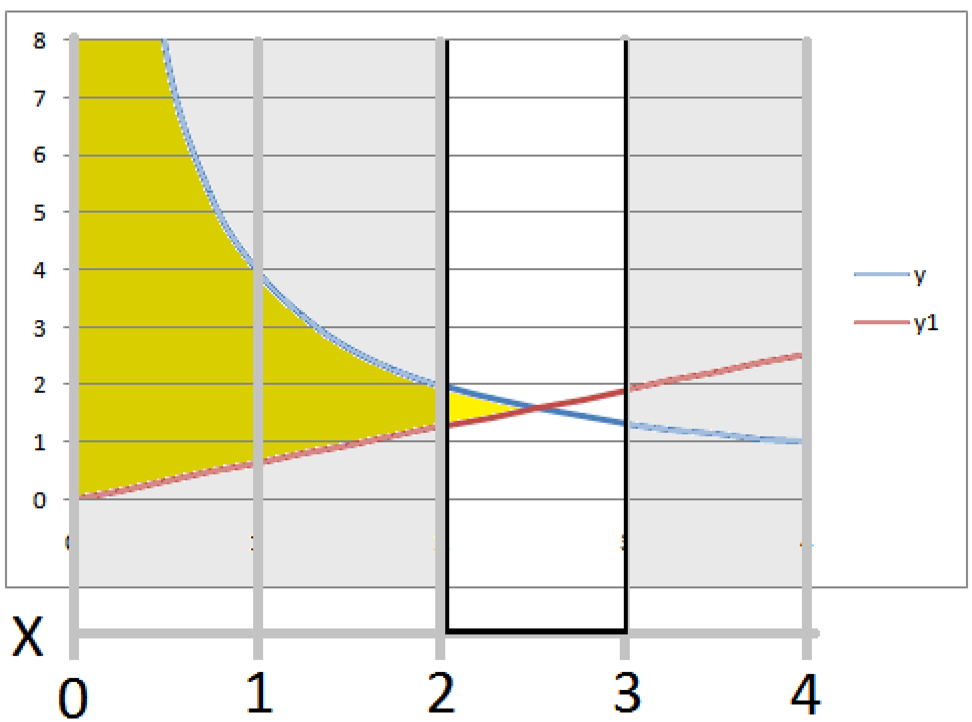
The optimum can be visually seen when O= -4X-Y is graphed for varying values of O.

From the graph, the upper bound can be found to be at -11.6

Discretization of a Variable: X

$$\sum_{d=1}^{D-1} X_d V_d \le X \le \sum_{d=1}^{D-1} X_{d+1} V_d$$

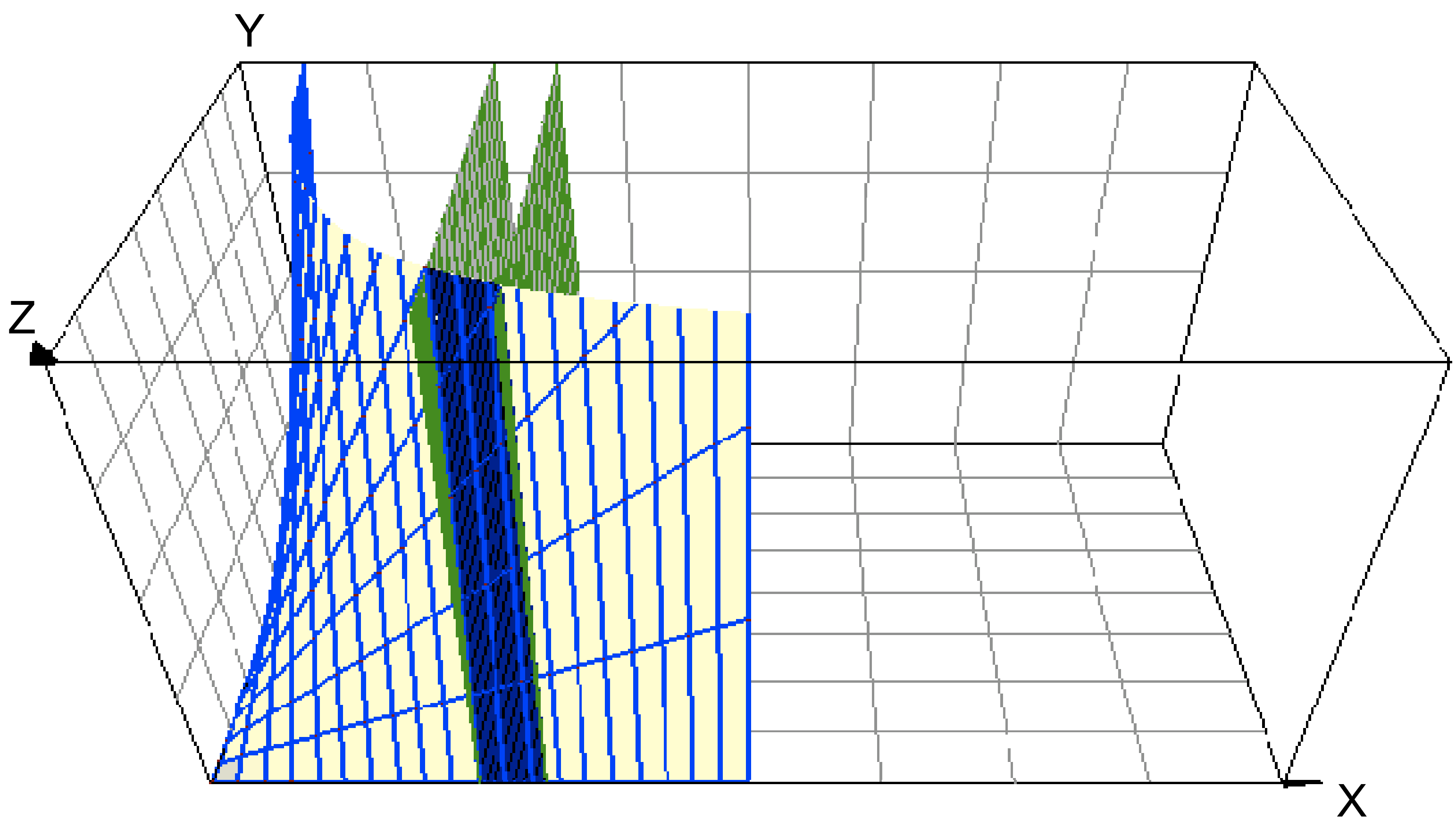
$V_d = binary\ variable \qquad \sum V_d = 1$



$d = 3$

Now to deal with the issue of Non-Convexity…

One of the slices, from x [2,3]



Now to deal with the issue of Non-Convexity…

$$\sum_{d=1}^{D-1} \widehat{X}_d V_d \leq X \leq \sum_{d=1}^{D-1} \hat{X}_{d+1} V_d$$

$$Z = X \cdot Y \leq 4$$

$$Z \geq Y \sum_{d=1}^{D-1} \hat{X}_d V_d$$

$$Z \leq Y \sum_{d=1}^{D-1} \hat{X}_{d+1} V_d$$

These terms are still non-linear

$$Z \leq 4$$

$$\sum_{d=1}^{D-1} \widehat{X}_d V_d \leq X \leq \sum_{d=1}^{D-1} \hat{X}_{d+1} V_d$$

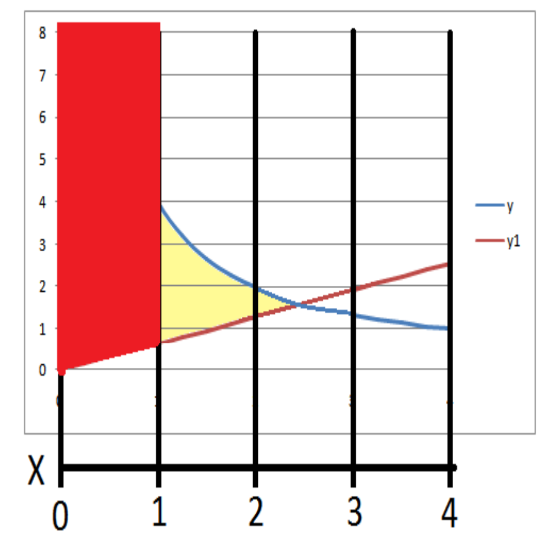
From the previous equations the feasible region when

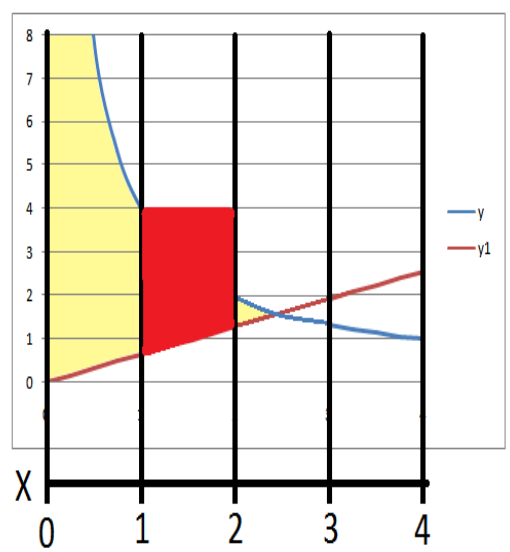
$$Z = X \cdot Y \leq 4 \qquad Z \geq Y \sum_{d=1}^{D-1} \hat{X}_d V_d \qquad Z \leq Y \sum_{d=1}^{D-1} \hat{X}_{d+1} V_d$$

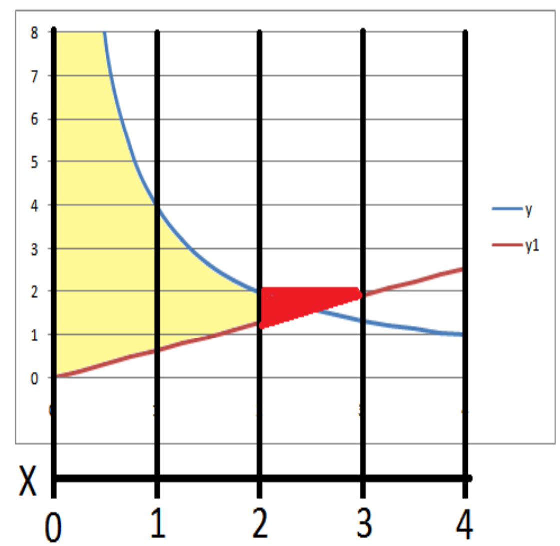
$$d = 1 \quad \left.\begin{array}{l} Z \leq 4 \\ Z \geq 0 \\ Z \leq y \end{array}\right\} \; 0 \leq y \leq 8$$

$$d = 2 \quad \left.\begin{array}{l} Z \leq 4 \\ Z \geq y \\ Z \leq 2y \end{array}\right\} \; 0 \leq y \leq 4$$

$$d = 3 \quad \left.\begin{array}{l} Z \leq 4 \\ Z \geq 2y \\ Z \leq 3y \end{array}\right\} \; 0 \leq y \leq 2$$







Fixing the integer × continuous variable nonlinearity

Defining: $W_d = Y \cdot V_d$

$$Z \leq Y \sum_{d=1}^{D-1} \hat{X}_{d+1} V_d$$

$$Z \geq \sum_{d=1}^{D-1} \hat{X}_d W_d$$

$$Z \leq \sum_{d=1}^{D-1} \hat{X}_{d+1} W_d$$

We get the following equations:

$$W_d \geq 0$$

$$W_d - Y^u \cdot V_d \leq 0$$

$$(Y - W_d) - Y^u(1 - V_d) \leq 0$$

$$(Y - W_d) \geq 0$$

V_d=0

$$W_d \geq 0, \quad W_d \leq 0 \quad \Rightarrow \quad W_d=0$$

$$(Y - W_d) - Y^u \leq 0, \quad (Y - W_d) \geq 0 \quad \Rightarrow \quad \text{Trivial}$$

V_d=1

$$W_d \geq 0, \quad W_d - Y^u \leq 0 \quad \Rightarrow \quad \text{Trivial}$$

$$(Y - W_d) \leq 0, \quad (Y - W_d) \geq 0 \quad \Rightarrow \quad W_d=Y$$

Rewritten Problem Statement

minimize O=-4X-Y

Subject To:

0≤X ≤4

0 ≤Y ≤8

$Z \leq 4$

$$Z \geq \sum_{d=1}^{D-1} \widehat{X}_d W_d$$

$$Z \leq \sum_{d=1}^{D-1} \hat{X}_{d+1} W_d$$

$$\sum_{d=1}^{D-1} \widehat{X}_d V_d \leq X \leq \sum_{d=1}^{D-1} \hat{X}_{d+1} V_d$$

$$W_d \geq 0$$

$$W_d - 8V_d \leq 0$$

$$(Y - W_d) - 8(1 - V_d) \leq 0$$

$$(Y - W_d) \geq 0$$

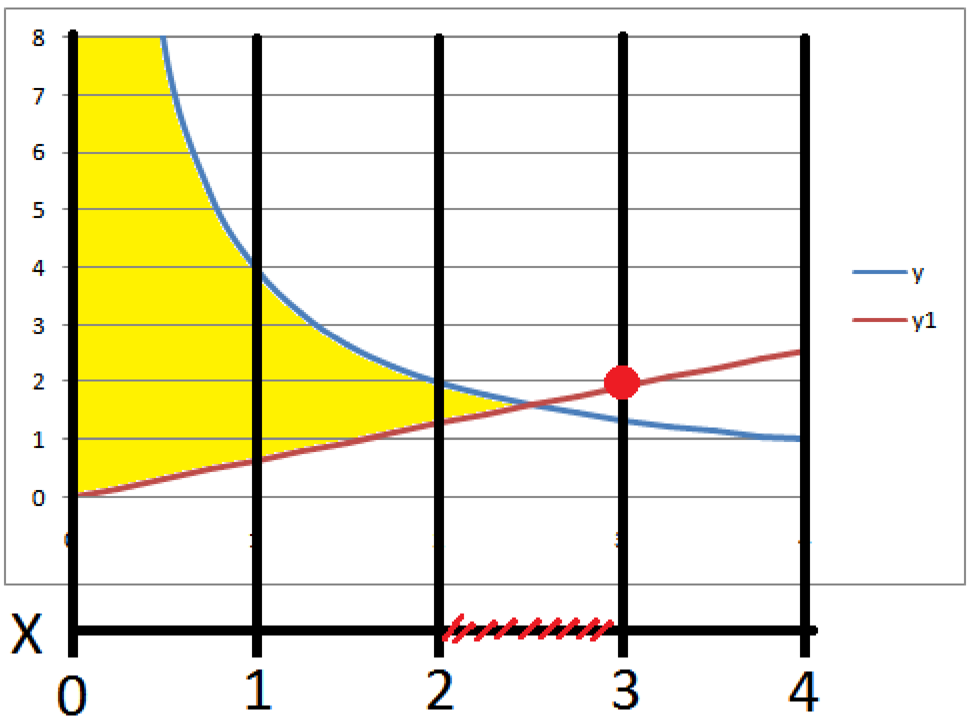
Method to solve for the Optimized function:

minimize O(-4X-Y)

$0 \leq X \leq 4$

$0 \leq Y \leq 8$

First solve for a lower bound using our method

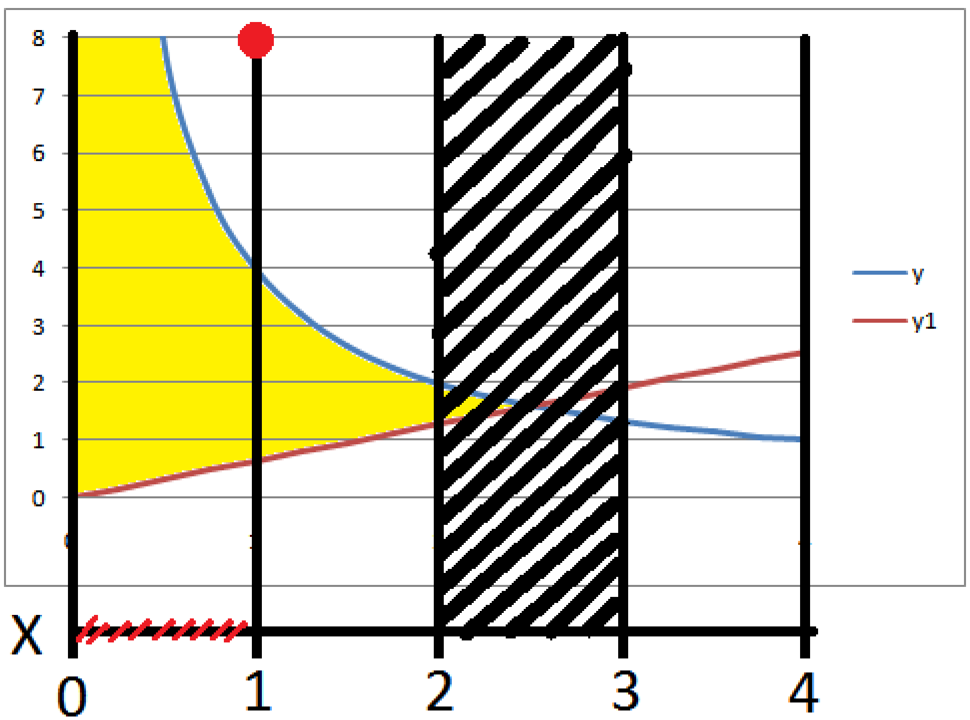


X=3

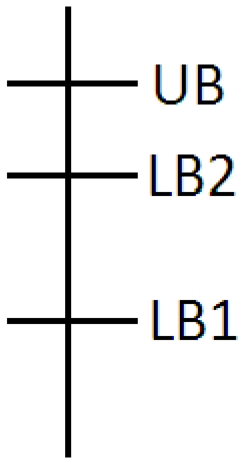
Y=2

L.B=-14

Now disregard the section field that was just used and solve again



X=1
Y=8
L.B.=-12



- Now we compare our new found lower bounds with an upper bound that has been previously found.

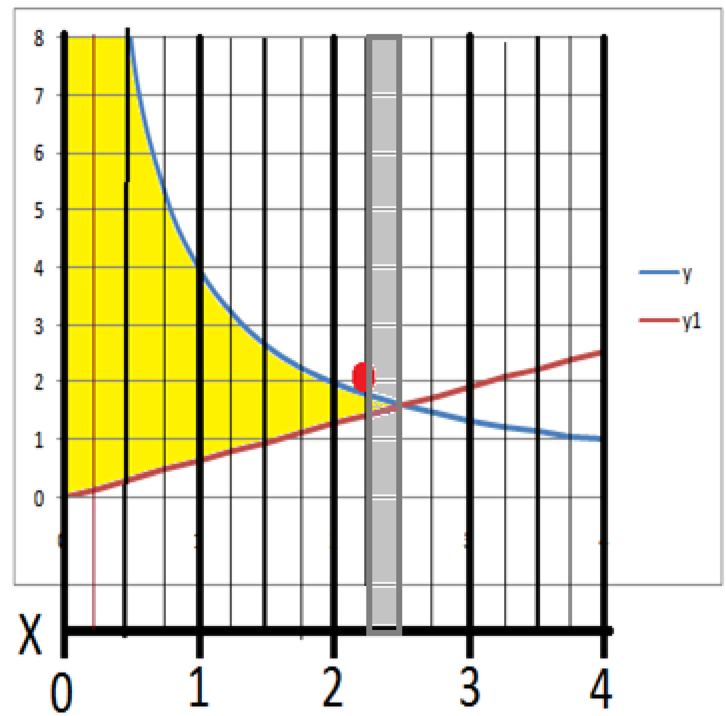
- If the second L.B is larger than the U.B, then we can disregard everything but the field that contained LB1.

- However, if we have a lower bound that is between, than we have two choices:
    - Branch and Bound on continuous domain
    - Cut the domain into more subunits and repeat the discretization method

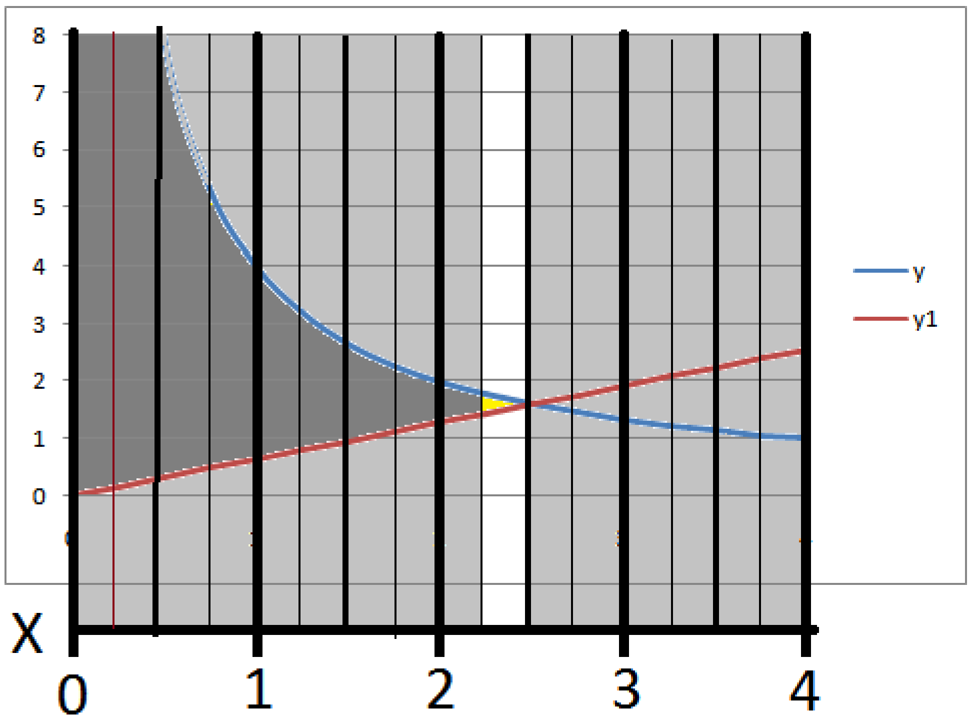
Continuing to Discretize the Variable more:

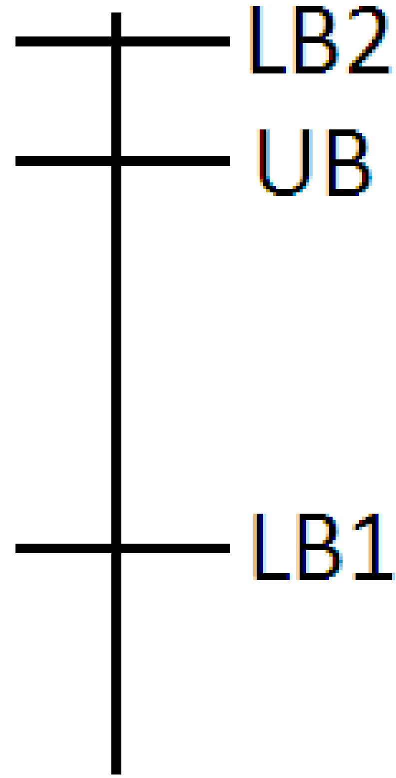
X=2.5
Y=1.78
L.B.=-11.78

X=2.25
Y=2
L.B=-11

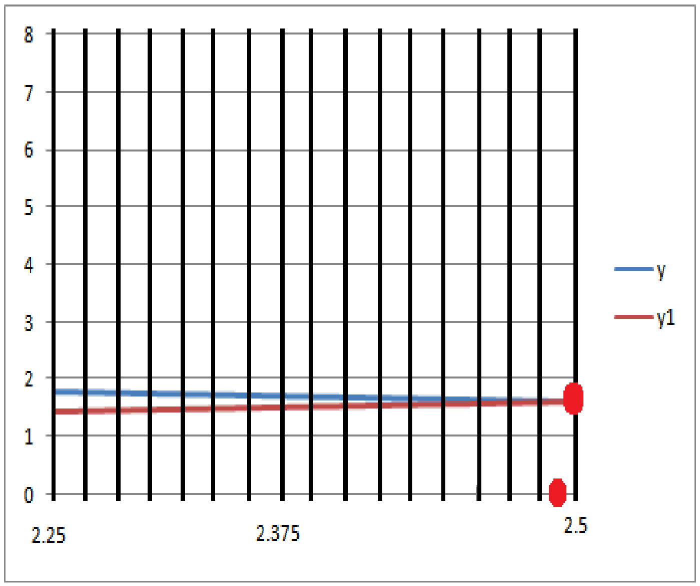


Since the New L.B. 2 > U.B, than we can disregard all the rest of the fields outside the location of L.B.1

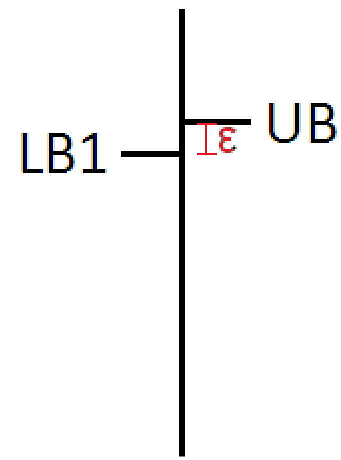




Solution:



X=2.5
Y=1.6
L.B.=-11.6



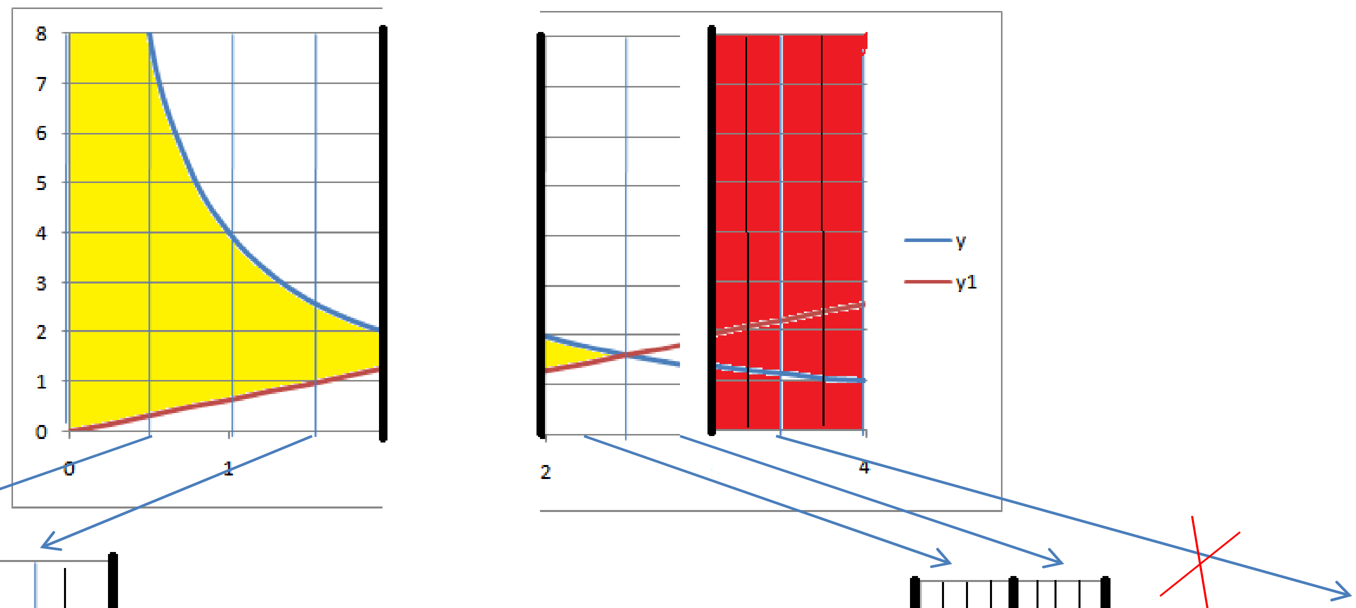
*When the LB>UB-ε than you have found the global Optimum

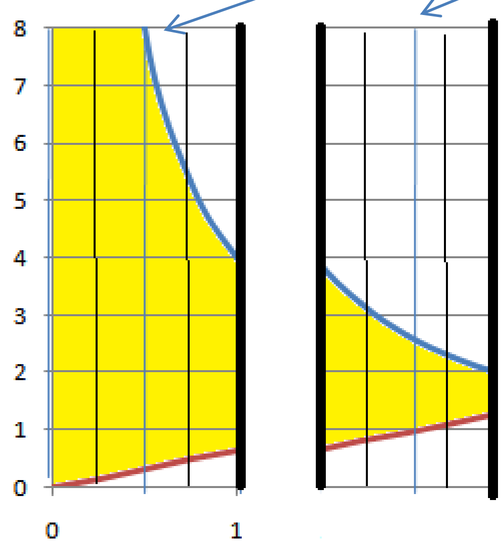
If there is no willingness to discretize further, move on to using the Branch and Bound Method.

Branch and Bound Method

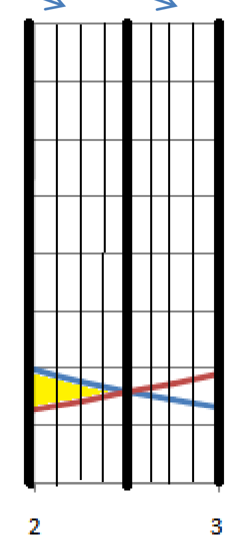
- Discretize in X, Bound Contract in X
- Discretize in Y, Bound Contract in Y
- Discretize in X, Bound Contract in Y
- Discretize in Y, Bound Contract in X

Branch and Bounding on the X Domain
With Contraction in the X Domain

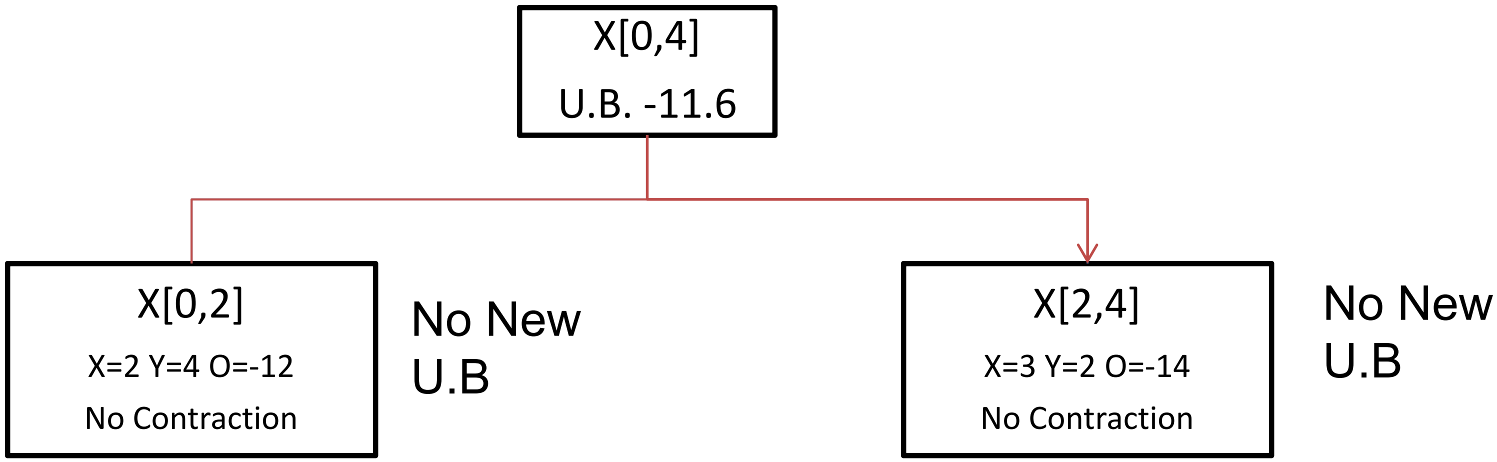




- Split up the domain
- Bound contract on the new domain
- Eliminate if possible

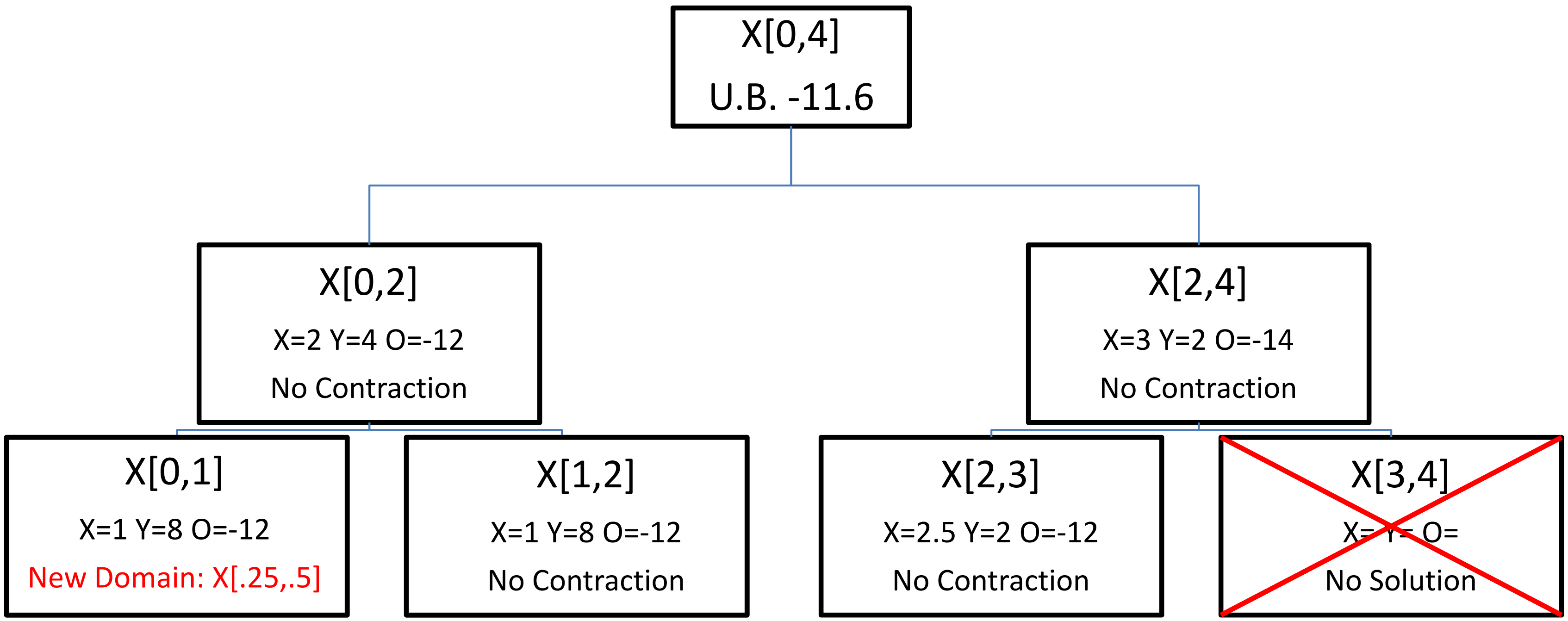


Branch and Bounding on the X Domain With Contraction in the X Domain



- Create the first branch => X[0,2] & X[2,4]
- Calculate the new U.B. with best solving method
- Contract within the domain

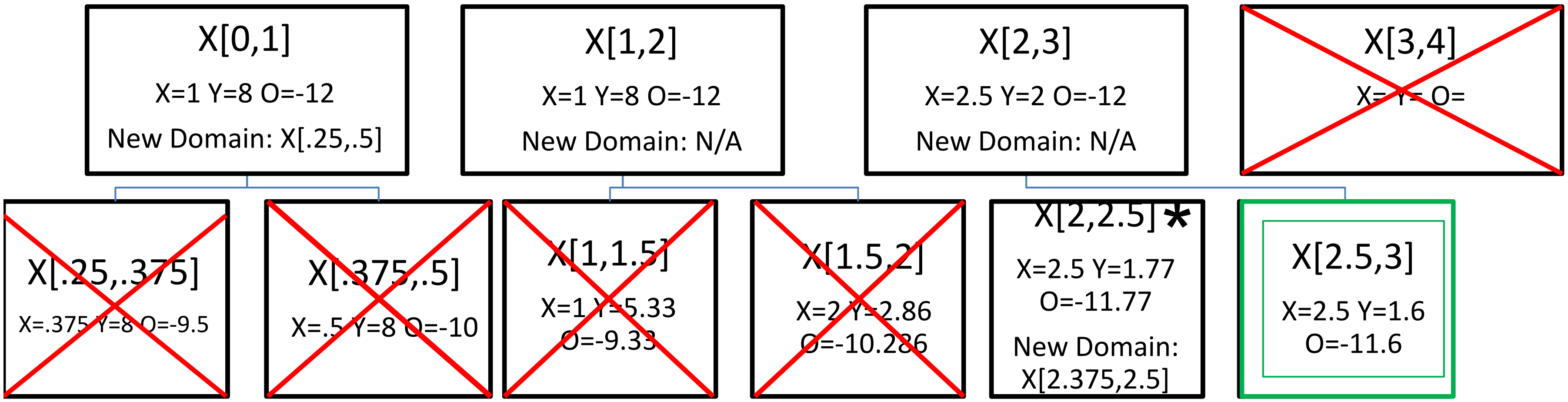
Branch and Bounding on the X Domain With Contraction in the X Domain



Continue Branching and Contracting, noting any new changes in the original Upper Bound

Look for places where either you can find a new domain or even cancel out a whole branch

Branch and Bounding on the X Domain
With Contraction in the X Domain

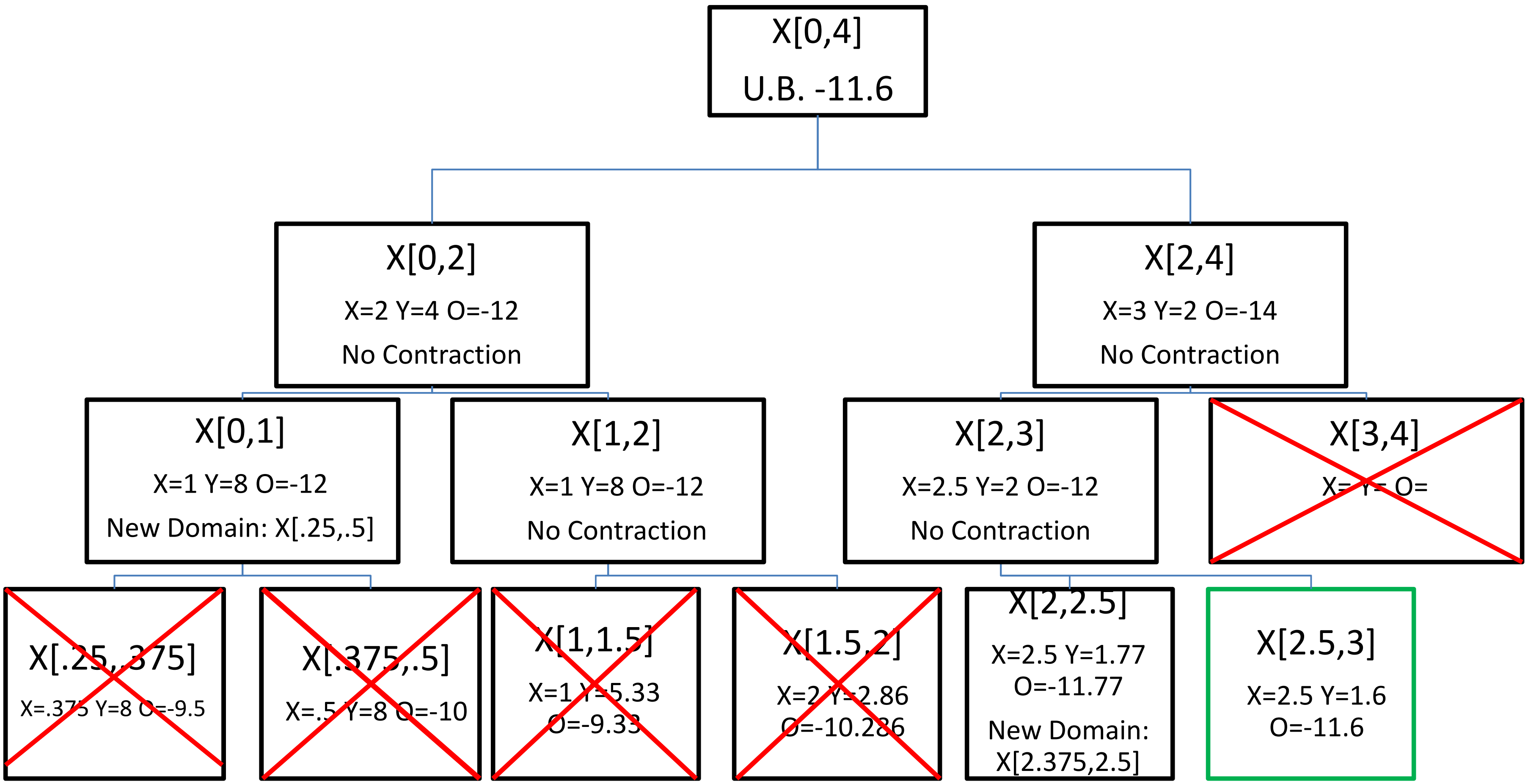


*this branch contracts toward X=2.5 as the number of branches increases

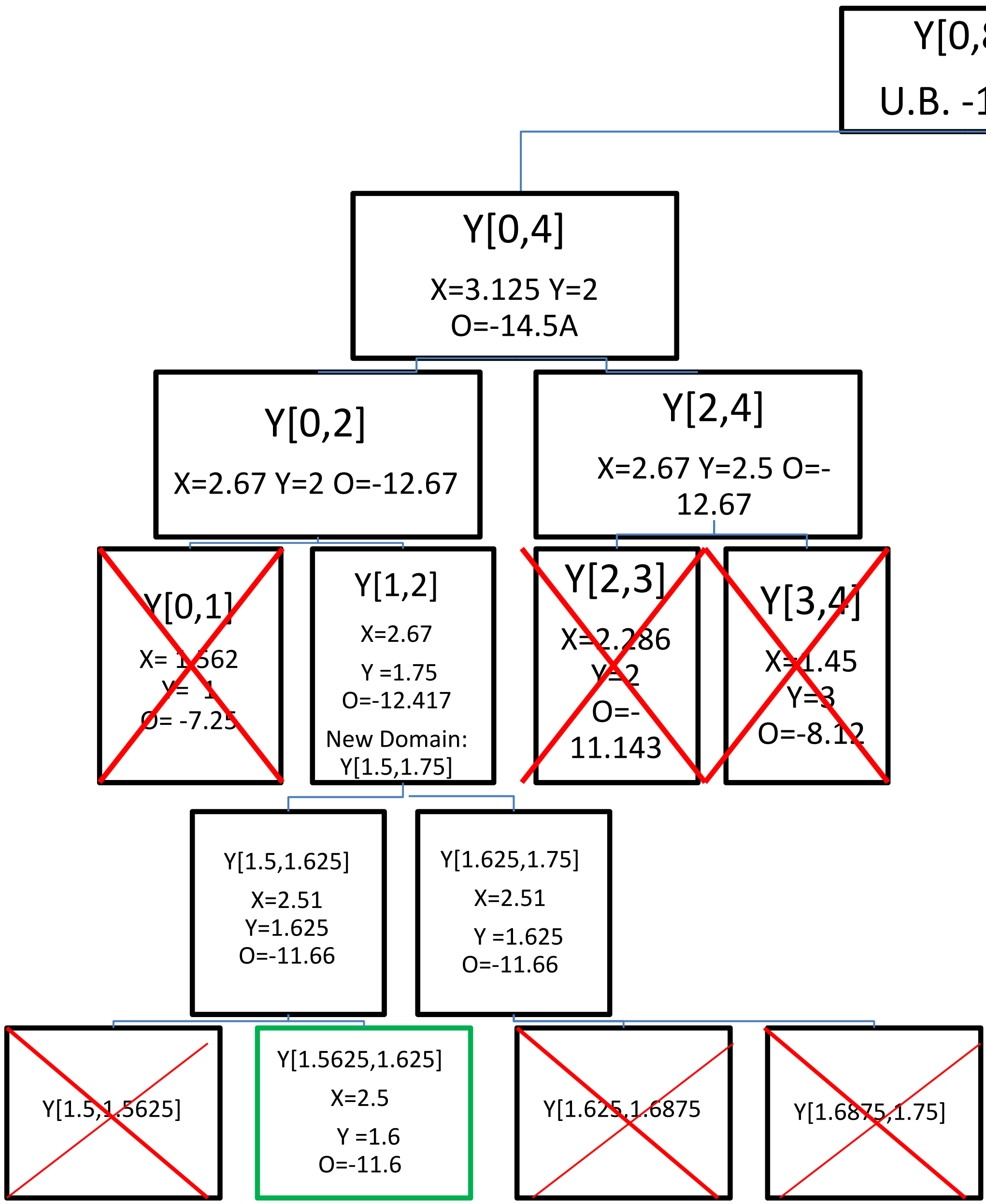
- Branch and Bound again canceling and finding new domains

- When a lower bound equals the upper bound, then the solution has been found

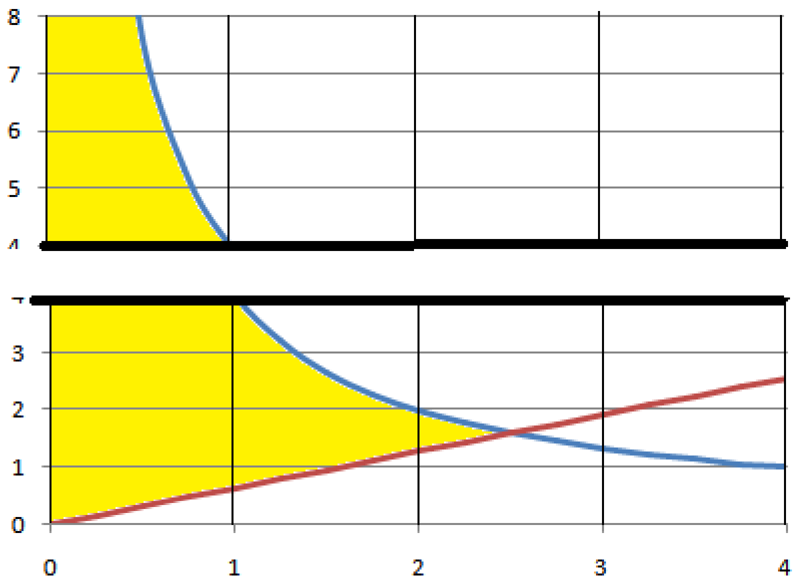
Branch and Bounding on the X Domain
With Contraction in the X Domain

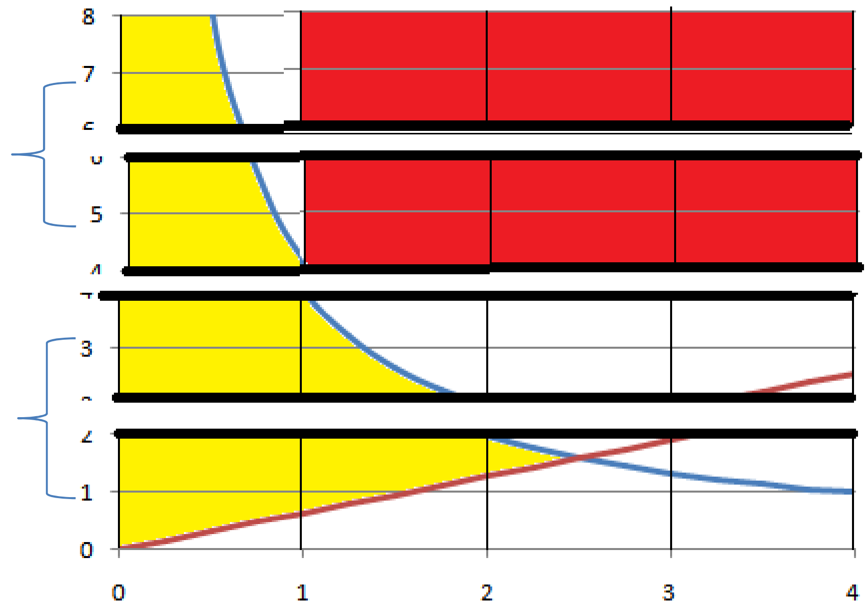


Branch and Bounding on the Y Domain
With Contraction in the Y Domain



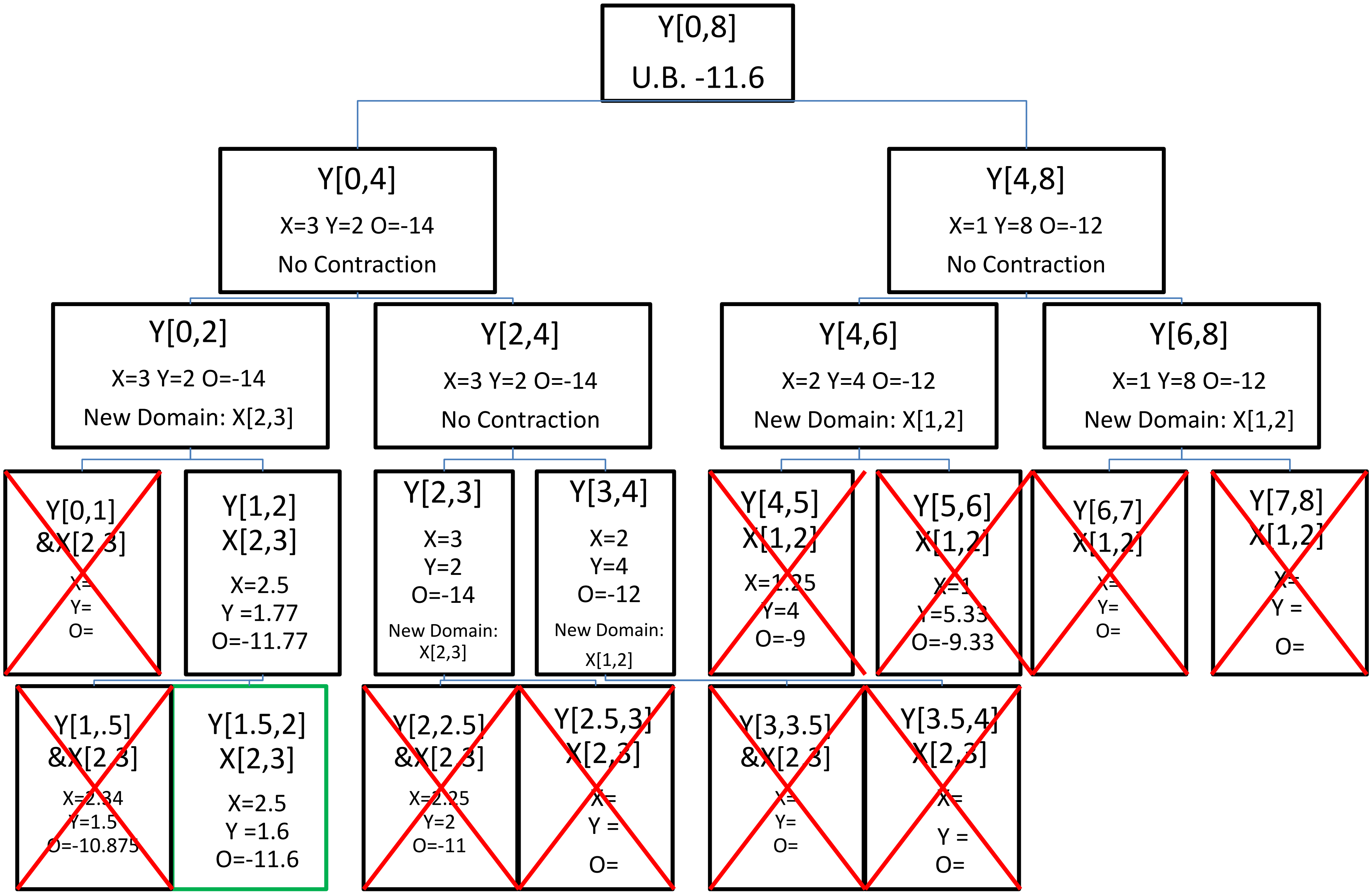
Branch and Bounding on the Y Domain
With Contraction in the X Domain



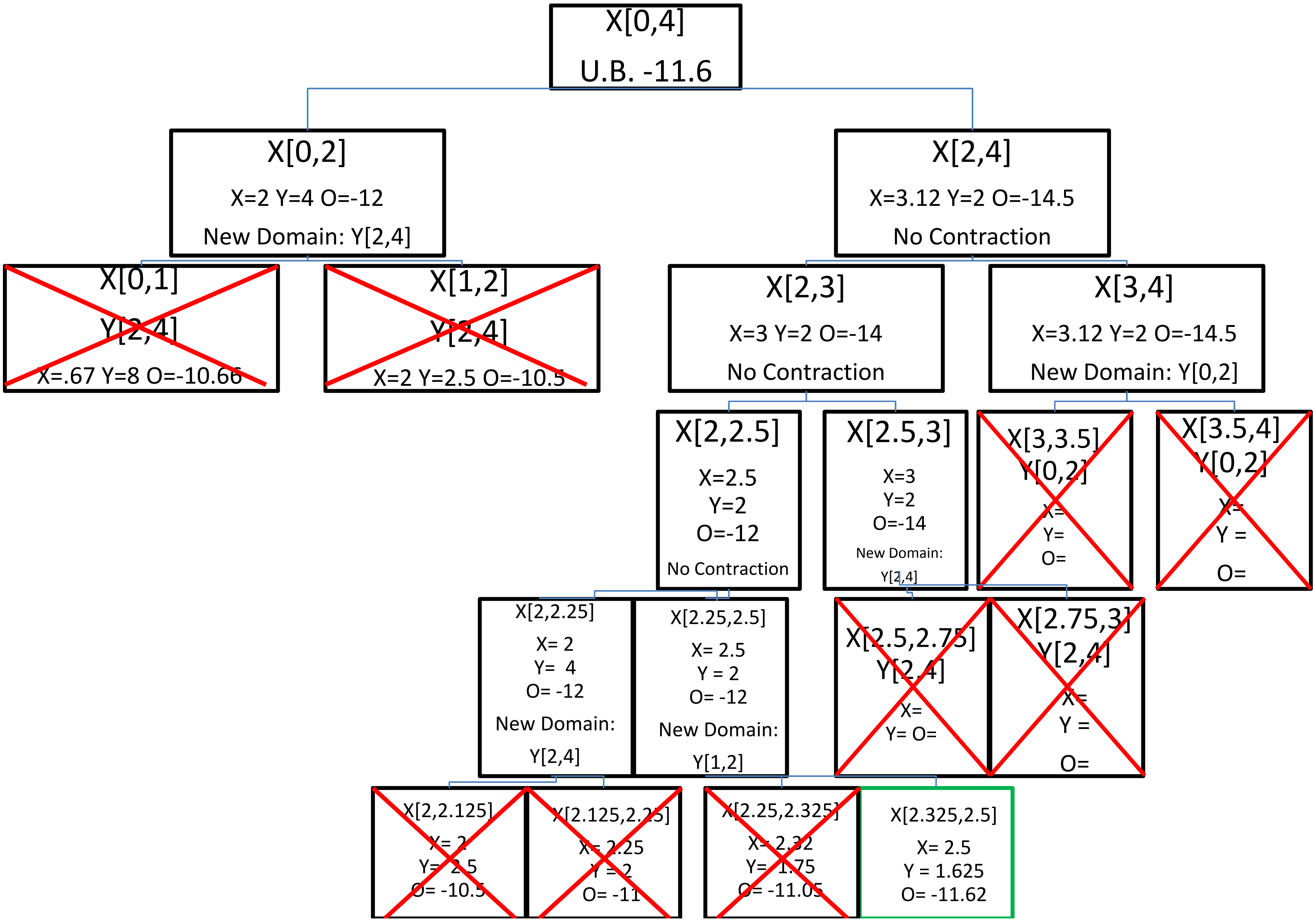


- Split up the domain
- Bound contract on the new domain
- Eliminate if possible

Branch and Bounding on the Y Domain With Contraction in the X Domain



Branch and Bounding on the X Domain With Contraction in the Y Domain



Which method is best?

•While all of the four techniques show great improvement over normal branch and bounding, it is difficult to predict which is best

•More research will have to be done to be able to predict or come up with a solid strategy to determine which method would be best to use

Multi-Variable Discretization

Similar to single variables, just more equations added:

- Discretize variables
- Get rid of nonlinearities

Different in solving technique options:

- Contracting on different variables
- Switching back and forth

New Problem:

$$minimize\ O = -3R - 3Y - X - P$$

$U.B = -8.75$

Variables:

$0 \leq P \leq 4$

$0 \leq R \leq 4$

$0 \leq X \leq 4$

$0 \leq Y \leq 8$

Subject To:

$R - .5P \leq 0$

$PR + X \leq 3 \qquad Z_0 = RP$

$Z_0 + X \leq 3$

$Y - .5X \leq 0$

$XY \leq 4 \qquad Z_1 = YX$

$Z_1 \leq 4$

Discretize Variables

Two Binary Variables needed, since there are two nonlinearities

$$0 \le P \le 4$$

$$V_0(d) = binary\ variable$$

$$\sum_{1}^{D-1} V_0 = 1$$

$$\sum_{1}^{D-1} V_0 \hat{P}_d \le P \le \sum_{1}^{D-1} V_0 \hat{P}_{d+1}$$

$$0 \le X \le 4$$

$$V_1(d) = binary\ variable$$

$$\sum_{1}^{D-1} V_1 = 1$$

$$\sum_{1}^{D-1} V_1 \hat{X}_d \le X \le \sum_{1}^{D-1} V_1 \hat{X}_{d+1}$$

Take care of Nonlinearities:

A Brief Refresher:

$$W_1 = V_1 Y$$

$$W_d \geq 0$$
$$W_d - Y^u \cdot V_d \leq 0$$
$$(Y - W_d) - Y^u(1 - V_d) \leq 0$$
$$(Y - W_d) \geq 0$$

$V_d=0$:

$$W_d \geq 0, \quad W_d \leq 0 \quad \Rightarrow \quad W_d=0$$

$$(Y - W_d) - Y^u \leq 0, \quad (Y - W_d) \geq 0 \quad \Rightarrow \quad \text{Trivial}$$

$V_d=1$:

$$W_d \geq 0, \quad W_d - Y^u \leq 0 \quad \Rightarrow \quad \text{Trivial}$$

$$(Y - W_d) \leq 0, \quad (Y - W_d) \geq 0 \quad \Rightarrow \quad W_d = Y$$

Rewriting the Nonlinearities in a linear way:

$$Z_0 = RP$$

$$\sum_{1}^{D-1} V_0 \hat{P}_d \le P \le \sum_{1}^{D-1} V_0 \hat{P}_{d+1} \qquad W_0 = R$$

$$\sum_{1}^{D-1} W_0 \hat{P}_d \le Z_0 \le \sum_{1}^{D-1} W_0 \hat{P}_{d+1}$$

$$Z_1 = YX$$

$$\sum_{1}^{D-1} V_1 \hat{X}_d \le X \le \sum_{1}^{D-1} V_1 \hat{X}_{d+1} \qquad W_1 = Y$$

$$\sum_{1}^{D-1} W_1 \hat{X}_d \le Z_1 \le \sum_{1}^{D-1} W_1 \hat{X}_{d+1}$$

Rewritten Problem Statement

$$minimize\ O = -3R - 3Y - X - P$$

Subject To:

$$0 \le P \le 4$$

$$0 \le R \le 4$$

$$0 \le Y \le 8$$

$$0 \le X \le 4$$

$$Z_0 + X \le 3$$

$$Z_1 \le 4$$

$$\sum_{1}^{D-1} V_0 \hat{P}_d \le P \le \sum_{1}^{D-1} V_0 \hat{P}_{d+1}$$

$$\sum_{1}^{D-1} V_1 \hat{X}_d \le X \le \sum_{1}^{D-1} V_1 \hat{X}_{d+1}$$

$$\sum_{1}^{D-1} W_0 \hat{P}_d \le Z_0 \le \sum_{1}^{D-1} W_0 \hat{P}_{d+1}$$

$$\sum_{1}^{D-1} W_1 \hat{X}_d \le Z_1 \le \sum_{1}^{D-1} W_1 \hat{X}_{d+1}$$

$$W_0 \ge 0$$

$$W_0 - 8V_0 \le 0$$

$$(Y - W_0) - 8(1 - V_0) \le 0$$

$$(Y - W_0) \ge 0$$

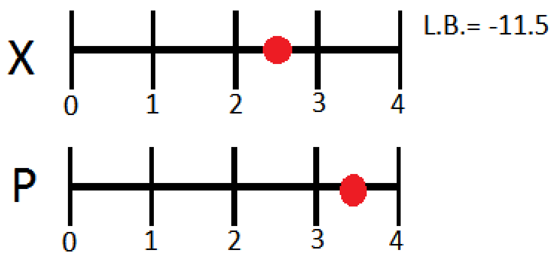
$$W_1 \ge 0$$

$$W_1 - 4V_1 \le 0$$

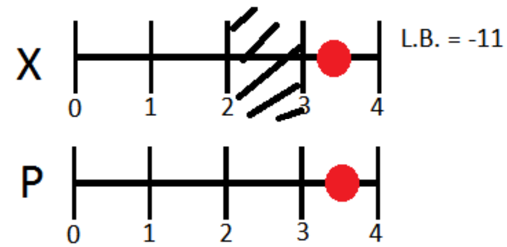
$$(R - W_1) - 4(1 - V_1) \le 0$$

$$(R - W_1) \ge 0$$

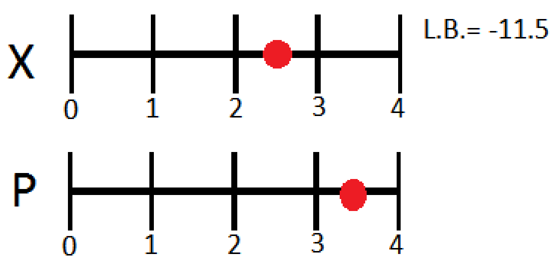
Solving the Problem: $U.B = -8.75$



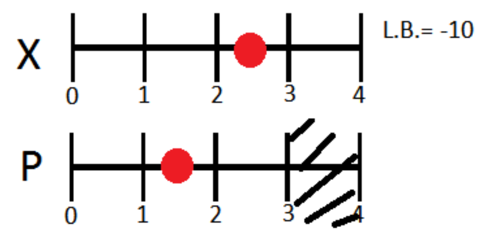
•Solve for the a lower bound
•Note the slices for this L.B.



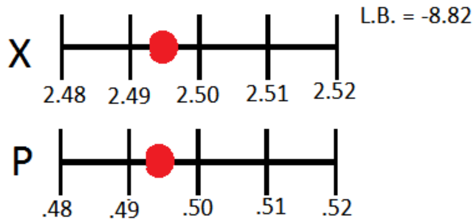
•Disregard the region in the x domain
•Solve for a new lower bound
•If L.B. > U.B. throw out other regions



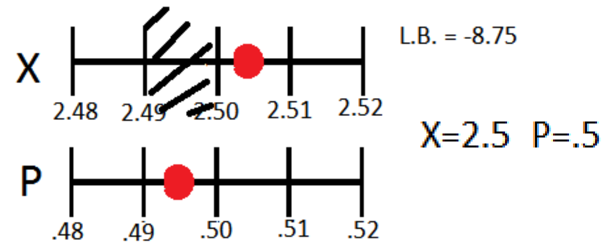
•If not solve try in the p domain



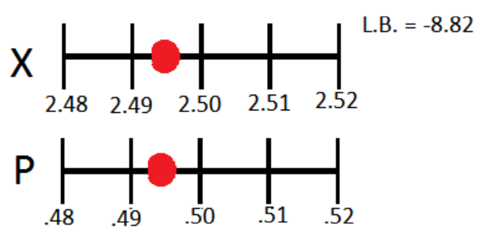
•Disregard the region in the p domain
•Solve for a new lower bound
•Since L.B.<U.B. discretize more



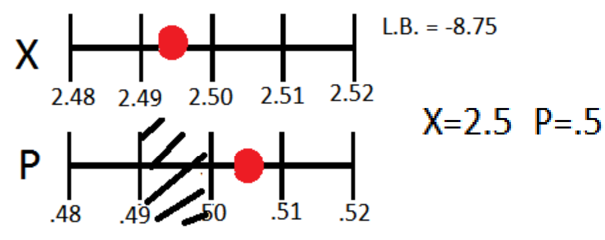
•With more discretizations, solve for a new L.B.



•Disregard the region in the x domain
•Solve for a new lower bound
•If L.B.= U.B.-ε the optimum is found



•Alternativly solving for the optimum with the p domain



•Similarly disregard the region in the p domain
•Solve for a new lower bound
•If L.B.= U.B.-ε the optimum is found

New options:

•Contract all the way on one variable until you can’t and then switch to the other

•Contract on one variable and then switch to the other, going back and forth

When solving optimization problems you have many options as to what you want to do.

The combinations will take more research to discover which is actually best, but currently it is a very promising method.